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Associate Minimal Surfaces.

BY JAMES K. WHITTEMORE.

It is a well-known fact, first proved by Schwarz,* that corresponding points of a family of associate minimal surfaces, corresponding in a sense presently to be explained, lie on an ellipse. In Part I of this paper we find the locus of the extremities of these ellipses, which we call Schwarz's ellipses, then find the envelope of a family of associate minimal surfaces and prove that the latter coincides with part of the locus named for real minimal surfaces applicable to surfaces of revolution, with certain exceptions; in Part II it is shown that this coincidence, with coincidence of corresponding points, occurs only when the minimal surfaces, supposed real, are applicable to a surface of revolution.

The Enneper-Weierstrass equations of a minimal surface S are

$$\left.\begin{aligned} x&=\frac{1}{2}\int(1-u^2)F(u)\,du+\frac{1}{2}\int(1-v^2)\phi(v)\,dv=U_1+V_1,\\ y&=\frac{i}{2}\int(1+u^2)F(u)\,du-\frac{i}{2}\int(1+v^2)\phi(v)\,dv=U_2+V_2,\\ z&=\int uF(u)\,du+\int v\phi(v)\,dv=U_3+V_3.\end{aligned}\right\}\quad(1)$$

The parameters u, v are the parameters of the minimal curves of S. When S is real F and ϕ are conjugate functions, and for a real point with a real tangent plane u, v have conjugate values.† The *adjoint* surface has the equations

$$x_1=i(U_1-V_1),\quad y_1=i(U_2-V_2),\quad z_1=i(U_3-V_3).$$

The equations of the associate minimal surface S_a are

$$\left.\begin{aligned} x_a&=U_1e^{ia}+V_1e^{-ia}=x\cos\alpha+x_1\sin\alpha,\\ y_a&=U_2e^{ia}+V_2e^{-ia}=y\cos\alpha+y_1\sin\alpha,\\ z_a&=U_3e^{ia}+V_3e^{-ia}=z\cos\alpha+z_1\sin\alpha.\end{aligned}\right\}\quad(2)$$

Values of α differing by $\pi/2$, substituted in (2), give adjoint surfaces; corresponding points of associate minimal surfaces are given by (2) when u, v are fixed and α varies. The locus of $x_\alpha, y_\alpha, z_\alpha$ for fixed u, v is Schwarz's ellipse. Points of the ellipse on adjoint surfaces are the extremities of conjugate diameters.*

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* H. A. Schwarz, "Miscellen aus dem Gebiete der Minimalflächen," *Journal de Crelle,* Vol. LXXX (1875).

† Eisenhart, "Differential Geometry," p. 256.

I.

§ 1. **The Locus L.** To find the extremities of the principal axes of the ellipse (2) we determine a point (α) such that the tangent at this point is perpendicular to the line joining it with the center. This condition gives

$$\Sigma(x\cos\alpha+x_1\sin\alpha)(-x\sin\alpha+x_1\cos\alpha)=0,$$

from which

$$\tan 2\alpha=\frac{2\Sigma xx_1}{\Sigma(x^2-x_1^2)}=i\,\frac{\Sigma(U_1^2-V_1^2)}{\Sigma(U_1^2+V_1^2)}.$$

The last equation determines two values of α, differing by $\pi/2$, unless $\Sigma U_1^2=\Sigma V_1^2=0$. The vanishing of these two sums is the condition that the ellipse be a circle. It may easily be proved that if Schwarz's ellipse is a circle for all points of a real minimal surface the latter is a plane. We find

$$\cos 2\alpha=\pm\frac{\Sigma(U_1^2+V_1^2)}{2\sqrt{\Sigma U_1^2\Sigma V_1^2}}.$$

Choosing the upper sign,

$$\cos\alpha=\pm\frac{1}{2}\,\frac{\sqrt{\Sigma U_1^2}+\sqrt{\Sigma V_1^2}}{\sqrt[4]{\Sigma U_1^2\Sigma V_1^2}},\quad \sin\alpha=\pm\frac{i}{2}\,\frac{\sqrt{\Sigma U_1^2}-\sqrt{\Sigma V_1^2}}{\sqrt[4]{\Sigma U_1^2\Sigma V_1^2}}.$$

Choosing again the upper signs, we have for one vertex, from (2),

$$x=U_1\sqrt[4]{\frac{\Sigma V_1^2}{\Sigma U_1^2}}+V_1\sqrt[4]{\frac{\Sigma U_1^2}{\Sigma V_1^2}},\tag{3}$$

with similar expressions for y and z. The two radicals in (3) are reciprocals, and conjugate for conjugate u, v. The other combinations of signs in the preceding equations give the other vertices of the ellipse; it appears that the four vertices are given by (3) by the four different determinations of the first radical. The locus L is the locus of the four vertices of Schwarz's ellipses, and its equations, in the parameters u, v, are given by (3) with the similar equations for y and z. The locus consists evidently of four nappes, symmetrical in pairs with respect to the origin; these symmetrical pairs we call L_1, L_2 and L_3, L_4. The squares of the semi-axes of the ellipse are given by $2(\Sigma U_1V_1\pm\sqrt{\Sigma U_1^2\Sigma V_1^2})$.

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* Scheffers, "Einführung in die Theorie der Flächen," 2d ed., p. 362.

Comparing equations (2) and (3) it is evident that the curve $\sqrt[4]{\Sigma V_1^2/\Sigma U_1^2}=e^{ia}$ is common to the two surfaces S_a and L_1; similarly, curves given by

$$\sqrt[4]{\frac{\Sigma V_1^2}{\Sigma U_1^2}}=-e^{ia},\quad \sqrt[4]{\frac{\Sigma V_1^2}{\Sigma U_1^2}}=-ie^{ia},\quad \sqrt[4]{\frac{\Sigma V_1^2}{\Sigma U_1^2}}=ie^{ia},$$

are common, respectively, to S_a and L_2, S_a and L_3, S_a and L_4; it is to be noted that the first of the last three equations, for example, is also the equation of a curve common to $S_{a+\pi}$ and L_1.

It may be proved that a part of the locus L coincides with one of the associate minimal surfaces only if the latter are plane.

§ 2. **The Envelope of S_a.** If we substitute in (2) the value of α in terms of u, v taken from the equation $|\partial x/\partial u \ \partial y/\partial v \ \partial z/\partial \alpha|=0$, we should expect to find both the envelope and the locus of singular points of the surfaces S_a, but it appears that only the envelope results, for the equation is an identity in α at the singular points. For the envelope it gives

$$e^{ia}=\pm\sqrt{\frac{(u+v)V_1+i(v-u)V_2+(uv-1)V_3}{(u+v)U_1+i(v-u)U_2+(uv-1)U_3}}.$$

Evidently the envelope consists of two nappes symmetrical with respect to the origin.

§ 3. **Minimal Surfaces Applicable to Surfaces of Revolution.** We apply the results of the preceding sections to families of real associate minimal surfaces applicable to surfaces of revolution.* All such surfaces are given by (1) where,

$$F(u)=cu^{m-2},\quad \phi(v)=\bar{c}v^{m-2},$$

m being a real constant, c and $\bar{c}$ conjugate constants. We call such surfaces B surfaces, as they were discovered by E. Bour, in particular a surface for any special value of m, not zero, B_m. It has been proved that all the surfaces associate to B_m are congruent, so that B_m is defined except for a homothetic transformation, and we may without restriction suppose c real; the associates of B_m are obtained by rotating the latter about the Z-axis; B_m and B_{-m} are congruent, so that we may suppose m positive; the curves of the surface, v/u constant, are geodesics and correspond to the meridians of the applicable surface of revolution; the curves, uv constant, are curves of constant total curvature, and correspond to the parallels of the surface of revolution. It is easily shown that the curves of B_m

$$\left(\frac{v}{u}\right)^m=1,\quad \left(\frac{v}{u}\right)^m=-1,$$

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* We have given an account of these surfaces and of the literature concerning them in a paper published in the *Annals of Mathematics*, Second Series, Vol. XIX, No. 1, September, 1917.

are respectively lines of curvature and asymptotic lines, and that the surface, when the constants of integration are taken as zero, is cut by the xy-plane in the latter. The value $m=0$ gives the minimal helicoids including the catenoid and the right helicoid; $m=2$ gives Enneper's surface.

Excluding $m=0, 1$, and taking all constants of integration as zero, we find

$$\sqrt[4]{\frac{\Sigma V_1^2}{\Sigma U_1^2}} = \sqrt{\frac{(u+v)V_1+i(v-u)V_2+(uv-1)V_3}{(u+v)U_1+i(v-u)U_2+(uv-1)U_3}} = \left(\frac{v}{u}\right)^{m/2}.$$

It follows that, for B_m, L_1 and L_2 coincide with the envelope of the associate surfaces. It is readily shown that L_1 is a surface of revolution whose axis is the Z-axis, and that L_3 is the xy-plane λ. If $m>1$, L_1 and L_2 form the locus of the extremities of the minor axes of the Schwarz ellipses, while L_3 and L_4, coinciding with the xy-plane, contain the major axes; if $m<1$, the situation is reversed. All ellipses corresponding to points of a curve of constant total curvature of B_m are equal. The curve previously mentioned, common to B_m and L_1 is $(v/u)^{m/2}=1$, which gives $y/x=\tan(2k\pi/m)$ where k is any integer; for B_m and L_2, $(v/u)^{m/2}=-1$ giving $y/x=\tan[(2k+1)\pi/m]$. The curves on L_1 are congruent plane curves, being meridians of the surface of revolution; they are lines of curvature and also geodesics of B_m, which is tangent to L_1 along each curve. The plane of each curve is a plane of symmetry of B_m. Similar statements apply to B_m and L_2. The curves common to B_m and L_3, B_m and L_4 are given respectively by

$$\left(\frac{v}{u}\right)^{m/2} = -i; \quad \frac{y}{x} = -\cot\frac{(2k+\frac{1}{2})\pi}{m}, \quad z=0,$$

$$\left(\frac{v}{u}\right)^{m/2} = i; \quad \frac{y}{x} = -\cot\frac{(2k-\frac{1}{2})\pi}{m}, \quad z=0,$$

and are straight lines in the xy-plane. Each line is a line of symmetry of B_m.

From the previous paragraph follow several theorems first proved by Ribaucour. A surface B_m, $|m|\neq 1$, has a number of congruent plane geodesic lines of curvature lying in planes which contain the Z-axis; half or all of these, depending on the value of m, are obtained by rotating one of them through successive angles $2\pi/m$, and an equal number are obtained by rotating about the Z-axis the symmetry of one of these curves with respect to the xy-plane through the angle π/m, then through successive angles $2\pi/m$; straight asymptotic lines of the surface, lying in the xy-plane, bisect the angles of the planes of the lines of curvature named above; if $m=p/q$, where p and q are integers with no common factor, and $q=1$ if m is an integer, the number of these plane lines of curvature is p and is equal to the number of straight asymptotic lines referred to; if m is irrational the number of each of these sorts of lines is infinite. Ribaucour's results are not so fully stated nor are they completely proved.

II.

§ 4. **Formulation of the Converse Problem.** The remainder of this paper is devoted to the investigation of the question: For what real minimal surfaces does the envelope of the associate surfaces coincide with part of the locus L, with coincidence of points given by the same u, v? We prove that the only surfaces having this property are the plane and B_m, $|m| \neq 1$, so that this coincidence is a characteristic property of real minimal surfaces applicable to surfaces of revolution. It must, however, be remarked that it does not hold for B_1 or for the minimal helicoids, $m=0$.

For the required coincidence we must have, for all u, v, either $U_1/V_1 = U_2/V_2 = U_3/V_3$ or

$$\frac{(u+v)V_1+i(v-u)V_2+(uv-1)V_3}{(u+v)U_1+i(v-u)U_2+(uv-1)U_3} = \pm\frac{\sqrt{\Sigma V_1^2}}{\sqrt{\Sigma U_1^2}}. \tag{4}$$

where we may assume without restriction that the radicals in (4) are conjugate for conjugate u, v. If the U's are proportional to the V's all six of these functions are constants so that the surface (1) is a point; this condition is also included in (4) so that the latter is the necessary and sufficient condition for the required coincidence.

Choosing first the upper sign in (4) we rewrite it in the form

$$\frac{U_1+iU_2+uU_3}{\sqrt{\Sigma U_1^2}}v+\frac{uU_1-iuU_2-U_3}{\sqrt{\Sigma U_1^2}} = \frac{V_1-iV_2+vV_3}{\sqrt{\Sigma V_1^2}}u+\frac{vV_1+ivV_2-V_3}{\sqrt{\Sigma V_1^2}}. \tag{4'}$$

Since (4′) is an identity in u, v the first number is linear in u, and as there can be no cancellation in its two terms we have

$$\frac{U_1+iU_2+uU_3}{\sqrt{\Sigma U_1^2}} = au+b, \quad \frac{uU_1-iuU_2-U_3}{\sqrt{\Sigma U_1^2}} = c+du, \tag{5}$$

where a, b, c, d are constants. From (4′) it follows that a and c are real, b and d are conjugate. If we take the lower sign in (4) we are led again to (5), but in this case a and c are pure imaginaries, b and $-d$ are conjugate.

The following discussion consists in the study of (5), to which we add the equations, implied in (1), $\Sigma U_1'^2=0$, and

$$U_1'+iU_2'+uU_3'=0. \tag{6}$$

Before undertaking the solution of these equations for U_1, U_2, U_3 we transform (5), writing

$$\sqrt{\Sigma U_1^2}=\rho, \quad U_1=\rho\lambda, \quad U_2=\rho\mu, \quad U_1=\rho\nu,$$

where, evidently,

$$\Sigma\lambda^2=\lambda^2+\mu^2+\nu^2=1, \quad \Sigma\lambda\lambda'=0.$$

From $\Sigma U_1'^2=0$,

$$\rho^2\Sigma\lambda'^2+\rho'^2=0, \quad \rho=e^{\pm i\int\sqrt{\Sigma\lambda'^2}du}.$$

Equations (5) may now be written

$$\lambda+i\mu=u(a-\nu)+b, \quad \lambda-i\mu=\frac{1}{u}(c+\nu)+d. \tag{5'}$$

Our problem may now be regarded as consisting of the solution of (5′) with $\Sigma\lambda^2=1$ for λ, μ, ν, then the determination of ρ and hence U_1, U_2, U_3, to be followed by applying (6) as a necessary condition.

We multiply together equations (5′) and replace $\Sigma\lambda^2$ by unity, finding

$$\nu=\frac{1-ac-bd-adu-bc/u}{a-c-du+b/u}.$$

Differentiating (5′) and multiplying together the resulting equations,

$$\Sigma\lambda'^2=\nu'(a+c)/u-[ac+\nu(a-c)-\nu^2]/u^2.$$

§ 5. **Solution for $b=d=0$, $a\neq c$.** For B_m we find $b=d=0$; it is therefore natural to attempt first the solution of the equations of § 4 on the assumption, $b=d=0$. We suppose further $a\neq c$. We find

$$\nu=\frac{1-ac}{a-c}, \quad \Sigma\lambda'^2=\frac{(1-a^2)(1-c^2)}{u^2(a-c)^2},$$

from which

$$\rho=Au^{\pm i\frac{\sqrt{(1-a^2)(1-c^2)}}{a-c}},$$

where A is an arbitrary constant. The values of λ and μ are found from (5′), and from them and the given values of ν and ρ we obtain U_1, U_2, U_3. We apply to the latter the condition (6); it appears that $a^2=1$ or $a^2c^2=1$. If a^2 or $ac=1$ the surface (1) is plane. It remains to consider only $ac=-1$. Replacing c by $-1/a$ we have two sets of values for the U's corresponding to the double sign in ρ; one of these satisfies (6) only if $a^2=1$, and may be discarded; the other satisfies (6) identically, and is

$$U_1=-\frac{Aa}{2}\,\frac{1-a^2}{1+a^2}\left[u^{\frac{2}{1+a^2}}+\frac{1}{a^2}u^{-\frac{2a^2}{1+a^2}}\right],$$

$$U_2=\frac{Aai}{2}\,\frac{1-a^2}{1+a^2}\left[u^{\frac{2}{1+a^2}}-\frac{1}{a^2}u^{-\frac{2a^2}{1+a^2}}\right],$$

$$U_3=\frac{2Aa}{1+a^2}\,u^{\frac{1-a^2}{1+a^2}}.$$

These give the equations for B_m if we set

$$a=\pm i\sqrt{\frac{m-1}{m+1}}, \quad A=\mp\frac{ci}{m\sqrt{m^2-1}}.$$

The first of the last two equations gives a real finite value for m unless $a=\pm i$, but such values of a are inadmissible, for both give, from $ac=-1$, $a=c$, a case excluded from our present consideration. The absolute value of m is greater or less than one as a is pure imaginary or real, so that these two cases of B_m, which show many geometrical differences, are also distinguished by the nature of a. It is interesting to observe that $m=0, 1$ are given by $a^2=0, 1$, respectively. These are the values of m excluded in § 3; the value $a^2=0$ is now excluded since $ac=-1$; $a^2=1$ gives for (1) a plane.

§ 6. **Solution for $b=d=0$, $a=c$.** We prove that when $b=d=0$ and $a=c$ the surface (1) is plane. Replacing c by a and writing $b=d=0$ in (5′) we have

$$\lambda+i\mu=(a-\nu)u, \quad \lambda-i\mu=(a+\nu)/u,$$

from which $\Sigma\lambda^2=a^2=1$. We may suppose $a=1$, for changing the sign of a merely changes the signs of λ, μ, ν and hence the signs of x, y, z in (1). It is not now possible to determine ν as in § 4. We find

$$\Sigma\lambda'^2=(\nu^2+2u\nu'-1)/u^2,$$

and substitute this value in the expression for ρ of § 4. Further,

$$2U_1=[1/u+u+\nu(1/u-u)]\rho, \quad 2U_2=[1/u-u+\nu(1/u+u)]\rho i, \quad U_3=\nu\rho;$$

substituting these and the value of ρ in (6),

$$\nu-1=\pm i\sqrt{\nu^2+2u\nu'-1},$$

from which $\nu=cu/(cu-1)$ and $\rho=c'(cu-1)/cu$, where c and c' are constants. Then $U_3=c'$ and (1) is the plane, $z=$constant.

§ 7. **Reduction of the General Case to $b=d=0$.** We now consider (5′) with no hypothesis concerning a, b, c, d. There are two cases: (I) a and c real, b and d conjugate, (II) a and c pure imaginary, b and $-d$ conjugate.

Since equations (5′) involve only u we may regard their transformation as that of the minimal curve,

$$x=U_1, \quad y=U_2, \quad z=U_3.$$

We prove in this section that this curve may be obtained by solving the equations of the preceding section after making a suitable linear transformation of u and a certain rotation of the coordinate axes to which the curve is referred. In the following section we show that the rotation is real, thereby proving that the surface found is unchanged by the transformation.

For any point of the curve the value of ρ is unaffected by a change of variable or by a rotation of the axes. Supposing the new equations of the curve to be

$$x_1=\rho\lambda_1, \quad y_1=\rho\mu_1, \quad z_1=\rho\nu_1,$$

we have $\lambda_1=\alpha_1\lambda+\beta_1\mu+\gamma_1\nu$, with similar equations for μ_1 and ν_1, where the coefficients are the terms of an orthogonal substitution. We write

$$u=\frac{\alpha u_1+\beta}{\gamma u_1+\delta},$$

and prove that by a suitable choice of this substitution and of the rotation (5′) becomes

$$\lambda_1+i\mu_1+u_1\nu_1=a_1u_1, \quad u_1(\lambda_1-i\mu_1)-\nu_1=c_1, \tag{7}$$

where a_1 and c_1 are both real or both pure imaginary, and $a_1c_1=-1$. Substituting for u in (5′) and clearing of fractions,

$$u_1[\gamma(\lambda+i\mu)+\alpha\nu]+\delta(\lambda+i\mu)+\beta\nu=u_1(a\alpha+b\gamma)+a\beta+b\delta,$$
$$u_1[\alpha(\lambda-i\mu)-\gamma\nu]+\beta(\lambda-i\mu)-\delta\nu=u_1(c\gamma+d\alpha)+c\delta+d\beta.$$

We make two combinations of the last equations, for the first multiplying the first equation by l, the second by l', and adding; for the second using m and m' as multipliers. We require that the first members of the equations so formed be respectively the first numbers of (7). This condition gives us values for λ_1, μ_1, and two values for ν_1, linear in λ, μ, ν; equating like coefficients in the two values of ν_1,

$$l=-m\delta/\gamma, \quad l'=m\beta/\gamma, \quad m'=-m\alpha/\gamma.$$

Substituting these values in λ_1 and μ_1, then using $\Sigma\lambda_1^2=\Sigma\lambda^2=1$, we find

$$\gamma^2/m^2=(\alpha\delta-\beta\gamma)^2,$$

and choose $m=-\gamma/(\alpha\delta-\beta\gamma)$, and both pairs of multipliers are determined. A rotation has now been found so that for an arbitrary linear substitution for u the first members of the transformed (5′) have the desired form; the coefficients of the substitution may now be chosen to simplify the second members of the equations. Equations (5′) are now replaced by

$$\lambda_1+i\mu_1+u_1\nu_1=a_1u_1+b_1, \quad u_1(\lambda_1-i\mu_1)-\nu_1=c_1+d_1u_1,$$

where

$$\begin{aligned}
a_1\Delta&=\ \ a\alpha\delta+b\gamma\delta-c\beta\gamma-d\alpha\beta,\\
b_1\Delta&=\ \ a\beta\delta+b\delta^2-c\beta\delta-d\beta^2,\\
c_1\Delta&=-a\beta\gamma-b\gamma\delta+c\alpha\delta+d\alpha\beta,\\
d_1\Delta&=-a\alpha\gamma-b\gamma^2+c\alpha\gamma+d\alpha^2,\\
\Delta&=\ \ \alpha\delta-\beta\gamma.
\end{aligned}$$

From these values it appears that

$$a_1+c_1=a+c, \quad a_1c_1-b_1d_1=ac-bd.$$

We wish so to determine α, β, γ, δ that $b_1=d_1=0$. This requires, since we must have $\Delta\neq 0$, that β/δ and α/γ be different roots of the quadratic, $dx^2+(c-a)x-b=0$. That the roots of this equation are distinct follows immediately since the discriminant can not vanish when (I) a and c are real, b and d conjugate, or (II) a and c are pure imaginary, b and $-d$ conjugate. Calling the roots of the quadratic x_1, x_2, writing $\alpha=\gamma x_1$, supposing $\delta=1$, as we may if $d\neq 0$, we have

$$a_1=\frac{b+ax_1-cx_2-dx_1x_2}{x_1-x_2}, \quad c_1=\frac{-b-ax_2+cx_1+dx_1x_2}{x_1-x_2}, \quad u=\frac{\gamma x_1u_1+x_2}{\gamma u_1+1}.$$

The values of a_1 and c_1 are independent of γ, so that the surface satisfying the requirement of our problem is applicable to itself in an infinite number of ways. It will appear in the next section that the surface is unchanged by the transformation when γ depends on an arbitrary real parameter. It will also appear that the new Z-axis is independent of γ.

If we apply (6) to the general equations (5′) we find, after rather tedious algebraic work, that for all surfaces other than the plane this condition becomes $ac-bd=-1$. This condition might certainly have been foreseen as necessary, for we have observed that $ac-bd$ is invariant under the transformation, and $ac=-1$ if $b=d=0$.

The values of a_1 and c_1 may also be found from the equations,

$$a_1+c_1=a+b, \quad a_1c_1=-1.$$

These give two pairs of values which exchange a_1 and c_1, and correspond to the interchange of x_1 and x_2. These two solutions give to m in B_m values differing only in sign, and therefore lead to the same surface. It may easily be proved that a_1 and c_1 are both real or both pure imaginary with a and c.

§ 8. **Reality of the Rotation.** Evidently to bring the minimal curve,

$$x=V_1, \quad y=V_2, \quad z=V_3,$$

into a corresponding reduced form, v must be transformed by a substitution conjugate to that applied to u, and the coordinate axes subjected to the rotation conjugate to that employed in § 7. To prove that the surface (1) is not changed by these two rotations we show that they are real and therefore identical. This proof may be given in two ways, both of which we indicate. The coefficients of λ, μ, ν in the expressions for λ_1, μ_1, ν_1, mentioned in § 7, may be expressed in terms of x_1, x_2, γ. The coefficients of ν_1 do not contain γ, so that the new Z-axis is the same for all reducing transformations. It may be shown directly that if (I) a and c are real, b and d conjugate, or (II) a and c are pure imaginary, b and $-d$ conjugate, and if further $\gamma = \sqrt{x_2/x_1}\, e^{i\phi}$, where ϕ is an arbitrary real number, all nine coefficients are real, hence that the surface satisfying the requirement of our problem is a surface B_m, $|m| \neq 1$. It appears also that the coordinate axes to which B_m in its standard form is referred, depend on an arbitrary real parameter, agreeing with a property of the associate surfaces mentioned in § 3. A second method of proving the rotation real consists in showing the rotation identical with its conjugate for the value of γ given above. This may easily be done if we note that $-1/x_2$, $-1/x_1$, $\sqrt{x_2/x_1}\, e^{-i\phi}$ are conjugate respectively to x_1, x_2, $\sqrt{x_2/x_1}\, e^{i\phi}$.

It is interesting to note that the linear transformation connecting u and u_1 is a real rotation of the sphere of the complex variable u for the value of γ given, and also that the transformation connecting the spheres on which u, v and u_1, v_1 are respectively the parameters of the minimal lines is a real rotation.

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